

## Note

---

# Maximality of the cycle code of a graph

Patrick Solé\*

*School of Computer and Information Science, Syracuse University, Syracuse, NY 13244, USA*

Thomas Zaslavsky\*\*

*Department of Mathematical Sciences, State University of New York, University Center at Binghamton, Binghamton, NY 13902-6000, USA*

Received 18 December 1990

Revised 3 June 1992

### Abstract

The cycle code of a graph is the binary linear span of the characteristic vectors of circuits. We characterize the graphs whose cycle codes are maximal for the packing problem, based on characterizing the graphs whose girth is at least  $\frac{1}{2}(n-c)+1$  where  $n$  and  $c$  are the numbers of vertices and connected components.

The cycle code (usually called cycle space) of a graph  $\Gamma$  is the linear span over  $F_2$  of the characteristic vectors of circuits versus edges. The packing properties of these codes have been extensively investigated in the past without achieving much success [4, 5]. To compare well with algebraic codes these codes had, in general, to be augmented [5]. That cycle codes which are *maximal* codes, in the sense that adding any new code words will reduce the minimum distance  $d$ , are very rare can be inferred from the examples in [4, 5] and explains the poor packing properties.

However, the rarity of maximal cycle codes has never been proved. Here we do so by finding them all. It is easiest to do this by first solving the weaker problem of

\*Currently at CNRS, I3S, 250 rue Albert Einstein, Bât. 4, Sophia Antipolis, 06560 Valbonne, France.

\*\*Research supported by the National Science Foundation under grant DMS-8808239.

finding all graphs whose girth  $g$  is 'very large', namely,  $g \geq \frac{1}{2}(n-c)+1$  where  $n$  is the number of vertices and  $c$  is the number of connected components.

For general graph theory background, we refer to [1]. All our graphs are finite and undirected. Loops and multiple edges are allowed. By  $\Gamma$ , we denote a graph  $(V, E)$  with  $n$  vertices,  $m$  edges,  $c$  connected components, and girth  $g$ . It is well known that  $C(\Gamma)$  is an  $[m, m-n+c]$  code whose minimum distance equals  $g$ .  $C_n$  is a circuit of length  $n$ . A *bowtie*, written  $C \cup_v C'$ , is the union of two circuits having in common just a vertex  $v$ . By  $pK_2$ , we mean a graph consisting of  $p$  edges joining the same two vertices. A subdivision of  $pK_2$  is called a *multipath*, for instance a *triple path* (also *theta graph*) if  $p=3$ . We need some notation for a subdivided  $K_4$  graph. Let  $v_1, v_2, v_3, v_4$  be the vertices of the  $K_4$ . Edge  $v_i v_j$  is subdivided into a path of length  $l_{ij}$ . Let  $S_0$  be the set of edges  $v_i v_j$  for which  $l_{ij}$  is even and let  $\beta=0$  if  $|S_0|$  is even, 1 if it is odd. An *opposite pair* is a pair of vertex-disjoint edges of  $K_4$ , or the corresponding pair of lengths.

**Theorem 1.** *The cycle code of a graph  $\Gamma$  is maximal iff  $\Gamma$  with isolated vertices removed is one of the following:*

- (a) *A forest (including the empty graph).*
- (b) *A graph in which every edge is a loop.*
- (c) *A multiple nonloop edge,  $mK_2$ .*
- (d) *A triangle, possibly with multiple edges.*
- (e) *A circuit  $C_g$  where  $g \geq 3$ , together with at most  $\lfloor (g-1)/2 \rfloor$  isthmi.*
- (f) *A disjoint union  $C_g \cup C_g$  where  $g$  is odd and  $g \geq 3$ .*
- (g1) *A bowtie  $C_g \cup_v C_g$  where  $g$  is odd and  $g \geq 3$ .*
- (g2) *A bowtie  $C_g \cup_v C_g$ , where  $g \geq 3$  and  $g$  is not a multiple of 4, together with an edge joining one vertex on each circuit having maximum distance from the common vertex  $v$ .*
- (h) *A subgraph of  $K_5$  having at least eight edges.*
- (i) *A theta graph whose three paths have lengths  $l_1, l_2, l_3$  such that (if  $\varepsilon$  denotes the number of even paths)  $\max l_i \leq f := \frac{1}{2}(l_1 + l_2 + l_3 - |\varepsilon - 1|)$ , together with at most  $f$  isthmi.*
- (j) *A quadruple path, three paths having the same odd length  $l \geq 3$  and the fourth having length  $l-1, l$ , or  $l+1$ .*
- (k)  *$K_{3,3}$  with either no edge, one edge, or the three edges of a perfect matching subdivided once.*
- (l) *A subdivided  $K_4$  graph in which:*
  - (1) *each opposite pair of lengths has difference  $|l_{ij} - l_{hk}| \leq \beta$ ;*
  - (2) *each opposite pair of lengths has sum at most  $\frac{1}{2}(n+2+\beta)$ ; and*
  - (3) *either  $|S_0| \leq 1$ , or  $S_0$  is a pair of opposite edges or a circuit of length 4, or  $S_0$  is a star or path of three edges and the larger edges of each opposite pair form a triangle.*

**Proof.** An  $[m, k]$  code  $C$  is maximal iff its covering radius,

$$R := \max_{y \in \mathbb{F}_2^m} \min_{x \in C} d(x, y),$$

where  $d(x, y)$  denotes the Hamming distance, satisfies  $R \leq d-1$  (see [2, p. 328]). To translate this into graph-theoretic terms we define a  $T$ -join in  $\Gamma$ , for  $T \subseteq V$ , as an edge set which has odd degree at every vertex in  $T$  and even degree at every other vertex (see [6, Section 6.5]). Necessarily  $T$  has even cardinality. We denote by  $\tau(\Gamma, T)$  the smallest possible cardinality of a  $T$ -join. The fundamental observation of [7] is that

$$R(C(\Gamma)) = \tau(\Gamma) := \max \{ \tau(\Gamma, T) : T \subseteq V, |T| \text{ even} \}.$$

(For brevity we write  $\tau$  for  $R(C(\Gamma))$  henceforth. Note that  $\tau$  remains the same if loops and isolated vertices are deleted and if multiple edges are simplified to single edges.) Obviously,

$$\tau(\Gamma) \geq \left\lceil \frac{n}{2} \right\rceil \quad \text{if } \Gamma \text{ is connected.} \quad (1)$$

The maximality criterion  $R \leq d-1$  becomes

$$\tau \leq g-1. \quad (2)$$

Combining (1) and (2) demonstrates that maximality entails very large girth. Hence the first half of the proof is to determine the graphs with the latter property (see Theorem 2).

The solutions to that problem are then tested in (2) by calculating  $\tau$ . This is reasonably straightforward; we omit the details. One can employ evaluations of  $\tau$  in [7] and the following lemma. Let  $\Gamma_v$  be the graph obtained from  $\Gamma$  by contracting the edges incident with a vertex  $v$ .

**Lemma 1.**  $\tau(\Gamma) = \tau(\Gamma_v) + 1$  if  $v$  is a divalent vertex of  $\Gamma$  which is not incident with a loop or isthmus.

Lemma 1 follows immediately from Frank's main theorem [3].

That concludes the proof of Theorem 1.

We now classify the graphs with very large girth.

**Theorem 2.**  $\Gamma$  has very large girth if and only if, ignoring isolated vertices, it is one of the following:

- (a) A forest (including the empty graph).
- (b) A graph whose edges are loops.
- (c)  $mK_2, m_1K_2 \cup m_2K_2$  (disjoint union), or  $m_1K_2 \cup_v m_2K_2$ .
- (d)  $K_3$  with at least one multiple edge.
- (e)  $C_g$ , where  $g \geq 3$ , possibly together with at most  $g-1$  isthmi.
- (f) A disjoint union  $C_g \cup C_g$  where  $g \geq 3$ .

(g) A bowtie  $C_g \cup_v C_g$  where  $g \geq 3$ , possibly with an additional edge whose endpoints belong to different  $C_g$ 's and where either both are at distance  $\lfloor g/2 \rfloor$  from  $v$  in the bowtie, or else  $g$  is even, one endpoint is at distance  $g/2$  from  $v$ , and the other is at distance  $(g/2) - 1$ .

(h)  $K_{2,5}$ , or  $K_{3,4}$  or  $K_5$  with possibly up to two edges deleted.

(i) A triple path with path lengths  $l_1, l_2, l_3$  satisfying  $\max l_i \leq \frac{1}{2}(l_1 + l_2 + l_3)$ , possibly together with at most  $(l_1 + l_2 + l_3) - 2 \max l_i$  isthmi.

(j) A quadruple path in which three paths have length  $l \geq 2$  and the fourth has length  $l - 1, l$ , or  $l + 1$ , possibly together with one isthmus if all path lengths are equal.

(k)  $K_{3,3}$ , possibly with an isthmus adjoined, or  $K_{3,3}$  with either one edge, the three edges of a perfect matching, the edges of a quadrilateral and the nonadjacent edge, or all nine edges each subdivided into a path of length 2.

(l) A graph  $\Gamma$  containing a subgraph  $\Gamma_0$  which is a subdivision of  $K_4$  and satisfying

(1)  $\Gamma = \Gamma_0$ , or each opposite pair of lengths is equal and  $\Gamma = \Gamma_0 \cup$  isthmus;

(2) in each opposite pair of lengths, the absolute difference is 0 or 1 and the sum is at most  $(n + 3)/2$ ; and

(3) either every opposite pair of lengths is equal, exactly one is unequal, or every one is unequal and the longer edges in each form a triangle.

**Proof.** Let the excess girth of  $\Gamma$  be  $\Delta g := g - \frac{1}{2}(n - c) - 1$ . Any solution is obtained from an isthmus-free one  $\Gamma$  by adding up to  $2\Delta g$  isthmi. We can therefore assume for nontriviality that  $\Gamma$  is simple, has no isthmi or isolated vertices, and is not a circuit.

**Lemma 2.** Suppose  $\Gamma$  contains vertex-disjoint circuits. Then it has very large girth iff  $\Gamma = C_p \cup C_p$  (disjoint union) where  $p \geq 3$ . In this case  $\Delta g = 0$ .

**Lemma 3.** Suppose  $\Gamma$  contains a bowtie. Then it has very large girth iff  $\Gamma$  is of type (g) or (h) in Theorem 3.1. In that case  $\Delta g = 0$ .

The lemmas imply that any graph with very large girth, not already discovered, is 2-connected. Thus, we can add to our standing assumptions the hypothesis that  $\Gamma$  is 2-connected and contains no vertex-disjoint circuits or bowtie. It follows that  $\Gamma$  contains a theta graph. If  $\Gamma$  contains no subdivided  $K_4$ , it is clearly a multipath so we are in case (h), (i), or (j). If  $\Gamma$  contains a subdivision of  $K_4$  but not of  $K_{3,3}$ , it must be the subdivided  $K_4$ . It is then only slightly complicated to prove we are in case (l). If  $\Gamma$  contains a subdivided  $K_{3,3}$  it can only be a subdivision of  $K_{3,3}$ . Then it is straightforward to prove we are in case (k).

This concludes the proof of Theorem 2.

## Acknowledgement

We wish to thank a referee for pointing out a simplification in the proof of Theorem 2.

## References

- [1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Elsevier, New York, 1976).
- [2] G.D. Cohen, M.G. Karpovsky, H.F. Mattson Jr and J.R. Schatz, Covering radius – survey and recent results, *IEEE Trans. Inform. Theory* 31 (1985) 328–343.
- [3] A. Frank, Conservative weightings and ear-decompositions of graphs, *Combinatorica*, submitted.
- [4] S.L. Hakimi, Recent progress and new problems in applied graph theory, Vol. 2, 1966 *IEEE Region Six Conference Record*, Tucson, AZ, pp. 635–643. *IEEE Region Six Conf. Record Comm.*, [Tucson, Ariz. (?), 1966(?)].
- [5] S.L. Hakimi and J.G. Bredeson, Graph theoretic error-correcting codes, *IEEE Trans. Inform. Theory* 14 (1968) 584–591.
- [6] L. Lovász and M.D. Plummer, *Matching Theory*, Ann. Discrete Math., Vol. 29 (North-Holland, Amsterdam, 1986).
- [7] P. Solé and T. Zaslavsky, The covering radius of the cycle code of a graph, *Discrete Appl. Math.* 45 (1993) 63–70.